1. Let $X_1, \ldots, X_n$ be a random sample from a Beta distribution with the probability density function:

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1, \text{ and } \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) **(3 pts)** Prove that the maximum likelihood estimator (MLE) of θ is $\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log(X_i)}$.

(b) **(3 pts)** Prove that the Fisher information for a single observation is $I(\theta) = \frac{1}{\theta^2}$.

(c) **(3 pts)** Use parts (a) and (b) to find the an approximate $100(1-\alpha)\%$ confidence interval for θ.

Hint: The Fisher information is defined by $I(\theta) = E\left[\frac{\partial \log f(X;\theta)}{\partial \theta}\right]^2 = -E\left[\frac{\partial^2 \log f(X;\theta)}{\partial \theta \partial \theta}\right]$.

Solution:

(a) The likelihood function is given