1. Consider the experiment of choosing a real number at random in the interval $(0, 1)$, where for any subinterval $(a, b) \subset (0, 1)$, the probability function is defined as follows:

$$P[(a, b)] = b - a. \qquad (1)$$

(a) **(2 pts)** Consider the sequence of events $\{C_n\}$ where $C_n = (a - \frac{1}{2^n}, a + \frac{1}{2^n})$. Determine with a brief explanation whether $\{C_n\}$ is increasing or decreasing.

(b) **(2 pts)** Use part (a) to find $\bigcup_{n=1}^{\infty} C_n$, and $\bigcap_{n=1}^{\infty} C_n$.

(c) **(2 pts)** Use parts (a) and (b), the definition of the probability function (1), and the Continuity Theorem of Probability to prove that for a particular point $a \in (0, 1)$, we have $P[\{a\}] = 0$.

Solution:

(a) For every $n \in \mathbb{N}$, $\frac{1}{2^{n+1}} < \frac{1}{2^n}$, and hence $(a - \frac{1}{2^{n+1}}, a + \frac{1}{2^{n+1}}) \subset (a - \frac{1}{2^n}, a + \frac{1}{2^n})$. Hence $\{C_n\}$ is decreasing.