Q1 (10 points) Let $\{C_n\}$ be a decreasing sequence of events. Show that

$$\lim_{n\to\infty} P(C_n) = P\left(\lim_{n\to\infty} C_n\right) = P\left(\bigcap_{n=1}^{\infty} C_n\right)$$

Solution: First note that since $\{C_n\}$ is a decreasing sequence of events, then $\{C_n^c\}$ is an increasing sequence of events. Define $R_1 = C_1^c$, $R_n = C_n^c - C_{n-1}^c$ for $n = 2, 3, \ldots$. It follows that R_n are mutually exclusive (disjoint) and further $\bigcup_{n=1}^{\infty} R_n = \bigcup_{n=1}^{\infty} C_n^c$. Then by De Morgan's law and Axiom 3 of probability, we have

$$\begin{aligned} P\left(\bigcap_{n=1}^{\infty} C_n\right) &= 1 - P\left(\bigcup_{n=1}^{\infty} C_n^c\right) = 1 - P\left(\bigcup_{n=1}^{\infty} R_n\right) = 1 - \sum_{n=1}^{\infty} P(R_n) \\ &= 1 - \lim_{n\to\infty} \sum_{i=1}^{n} P(R_i) = 1 - \lim_{n\to\infty} \left\{ P(R_1) + \sum_{i=2}^{n} P(R_i) \right\} \\ &= 1 - \lim_{n\to\infty} P(C_n^c) \end{aligned}$$