A67 Winter 2024: Midterm 1 (solution)

1. Prove the following statement by way of contradiction:

$$\forall n, k, x_1, ..., x_k \in \mathbb{N}, \quad \left[(x_1 \cdot x_2 \cdot ... \cdot \cdot x_k) > n \Longrightarrow \exists i \in [k], x_i > n^{1/k} \right] .$$

In your proof, write down the assumption that you are making towards a contradiction, explicitly and fully. *(25 pts)*

Assume towards a contradiction that

$$\exists n, k, x_1, ..., x_k \in \mathbb{N}, \quad \left[(x_1 \cdot x_2 \cdot ... \cdot \cdot x_k) > n \wedge \forall i \in [k], x_i \leq n^{1/k} \right] .$$

Since for every $i \in [k]$ it holds that $x_i \leq n^{1/k}$, it follows that

$$(x_1 \cdot x_2 \cdot x_3 \cdot ... \cdot x_k) \leq (n^{1/k} \cdot n^{1/k} \cdot ... \cdot n^{1/k}) = (n^{1/k})^k \leq n .$$

This contradicts the assumption that $(x_1 \cdot x_2 \cdot x_3 \cdot ... \cdot x_k) > n$.

Hence, we deduce that for every $n, k, x_1, ..., x_k \in \mathbb{N}$, if $(x_1 \cdot x_2 \cdot ... \cdot \cdot x_k) > n$ then