Problem 1 Sec 4.6

#5 (a) $\forall x, y \in \mathbb{R}^n$, $\langle Qx, Qy \rangle = \langle x, Q^t(Qy) \rangle$ for any matrix Q.

($\Rightarrow$) Suppose Q is orthogonal. $\langle Qx, Qy \rangle = \langle x, (Q^tQ)y \rangle = \langle x, y \rangle$

↑ $Q^tQ = I$ because Q is orthogonal

($\Leftarrow$) Suppose $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$.

$\langle x, (Q^tQ)y \rangle = \langle Qx, Qy \rangle = \langle x, y \rangle \Rightarrow Q^tQ = I$

$\Rightarrow Q$ is orthogonal. ↑ by assumption

(b) Suppose Q is orthogonal. $\|Qx\|^2 = \langle Qx, Qx \rangle = \langle x, x \rangle = \|x\|^2$

↑ From (a)

$\Rightarrow$ Since $\|Qx\| \geq 0$ and $\|x\| \geq 0$, $\|Qx\| = \|x\|$

(c) Let A be an $n \times n$ matrix. Suppose $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

$\langle Ax, Ay \rangle = \frac{1}{4}\left(\|Ax + Ay\|^2 - \|Ax - Ay\|^2\right)$ by polarization identity

$= \frac{1}{4}\left(\|A(x+y)\|^2 - \|A(x-y)\|^2\right)$

$= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$ ↓ by the assumption

$= \langle x, y \rangle$