Problem 4. $[\,]_\alpha : V \to \mathbb{R}^n$ defined by $[x]_\alpha$, the coordinates of x with respect to α.

(Sol)

$\forall x, y \in V$, Say $\alpha = \{w_1, \cdots, w_n\}$. Since α is a basis for V,

$x = t_1 w_1 + \cdots + t_n w_n$ for some $t_i \in \mathbb{R}$ $1 \le i \le n$

$y = s_1 w_1 + \cdots + s_n w_n$ for some $s_i \in \mathbb{R}$, $1 \le i \le n$

$x+y = (t_1+s_1)w_1 + \cdots + (t_n+s_n)w_n$

$[x+y]_\alpha = (t_1+s_1, \cdots, t_n+s_n) = (t_1, \cdots, t_n) + (s_1, \cdots s_n)$

$= [x]_\alpha + [y]_\alpha$

For any $k \in \mathbb{R}$, $kx = kt_1 w_1 + \cdots + kt_n w_n$

So $[kx]_\alpha = (kt_1, \cdots, kt_n) = k(t_1, \cdots, t_n) = k[x]_\alpha$

So, $[\,]_\alpha$ is linear.