Problem 1

Sec 2.6 # 1(a) $T(p(x)) = xp'(x)$. Since $T(1) = x(1)' = x \cdot 0 = 0$ and $T(z) = x(z)' = x \cdot 0 = 0$,

$\ker(T) \neq \{0\}$. $\Rightarrow$ T is not injective

1(b) $[T] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is a REF. No free variable $\Rightarrow \ker(T) = \{0\} \Rightarrow T$ is injective

Since the dimensions of the domain and codomain are the same, T is surjective too

To find $[T]^{-1}$ solve $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2}$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ So $[T]^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Therefore, $T^{-1}(x_1, x_2, x_3) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1 + x_2 - x_3, -x_2 + x_3, x_3)$

1(c) $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(p(x)) = (p(0), p(1), p(2))$

Claim: T is injective.

Pf $\forall\, p(x) = a_0 + a_1 x + a_2 x^2 \in \ker(T)$