# APM236[1]
## Sample Problems and Solutions

(1) (a) Let $a$ be a (column) vector in $\mathbb{R}^n$. Show that the function $f(x) = a^T x$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$.

**Solution**

To check check $f$ is linear, recall the definition: a function is linear if for all $\alpha, \beta \in \mathbb{R}$ and for all $x, y \in \mathbb{R}^n$, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.
Now we start from one side and use the definition of $f$: $f(\alpha x + \beta y) = a^T(\alpha x + \beta y) = \alpha a^T x + \beta a^T y = \alpha f(x) + \beta f(y)$ which is precisely $\alpha f(x) + \beta f(y)$.

(b) Show that any linear function $f : \mathbb{R}^n \to \mathbb{R}$ is of the form $f(x) = a^T x$ for some vector $a \in \mathbb{R}^n$. (Hint: let $e_i$ denote the standard unit vectors in $\mathbb{R}^n$ and write $x$ as $x = x_1 e_1 + \cdots + e_n x_n$. Let $a_i := f(e_i) \in \mathbb{R}$. What is $f(x_1 e_1 + \cdots + e_n x_n)$?)